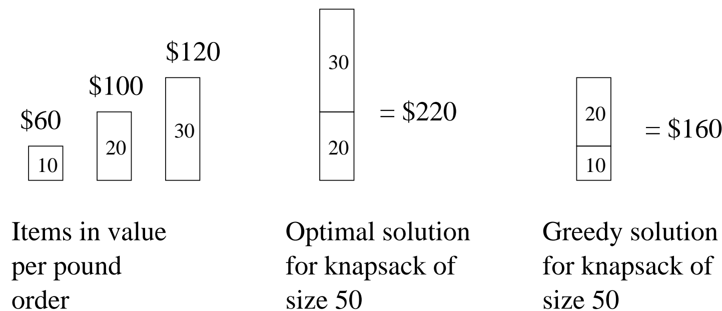
DP Example 3: The 0-1 Knapsack Problem

Module 4:Techniques

- **The $0-1$ knapsack problem**: We are given a knapsack of capacity W (that is, it can hold at most W pounds), which we can fill by choosing any subset of n items; for each item, we know its weight and value; these are given as two arrays, $v[]$ and $w[]$, where item i is worth $v[i]$ and has weight $w[i]$ pounds. The goal is to fill the knapsack in such a way so that the value of all items in the knapsack is maximized. Assume that weights $w[i]$ and the total weight W are integers.
- If this sounds a little abstract, imagine you are a thief in Alibaba's cave; you can't carry everything, so you want to fill your backpack with stuff in such a way that you maximize the value.
- Or perhaps, you pack for a camping trip. You can't take all the stuff you want, so you want to maximize the value you pack. You get the idea.

- How to decide which items to take and which ones to leave? That's the question.
- Straightfirward solution: Enumerate all possible subsets of items; calculate the total weight and the total value of each subset; find the max value among all subsets whose total weight is $< W$. Analysis: How many subsets in a set of n elements? It's 2^n. So the straightfirward solution is exponential.
- How can we improve this? We'll use dynamic programming.
- **Wait a minute. Would something a lot simpler work?** One very tempting idea is the following: Take the item with the largest value-per-pound, then the second one, and so on, while there is space left in the knapsack. This is called a *greedy algorithm*; Sometimes greedy strategies work (we'll talk more about greedy algorithms next week); but not in this case—below is a counter-example showing that the strategy above does not work for the knapsack problem:



- One can imagine a version of the problem called the FRACTIONAL-KNAPSACK PROBLEM in which we can take fractions of items. In this case, we would take $\frac{2}{3}$ of \$120 object and get \$240 solution. The counter-example above would not work anymore, and in fact we'll show that the fractional knapsack problem can be solved with a greedy strategy. More on this next week.

- **Optimal substructure**: Howcan we argue that the 0-1 knapsack problem has optimal substructure?

  Consider an optimal solution O, which is essentially a subset of items in $I = \{1, 2, ..., n\}$. We know that O is the maximal way to pack a knapsack of capacity W with items in I.

  Let i be an item in the optimal solution O. What can we say about the the remaining items in O, and the remaining items in I?

  Answer: $O - \{i\}$ must be the optimal way to pack a backpack of weight $W - w[i]$ with items in the set $I - \{i\}$.

  Why? By contradiction. If there was a better way to pack a backpack of weight $W - w[i]$ with items in $I - \{i\}$, then we could take that, add element i, and that would give us a solution that's better than O —- contradiction.

- **Simplify:** As usual, we'll focus on finding out the optimal value that we can place in the knapsck; At the end we'll come back and think how to augment the solution to find the actual set of items in addition to the value.

- **Recursive formulation:** The hardest part is coming up with the recursive formulation. We note that as we put an item in the knapsack, the set of remaining items to choose from is smaller, and the weight of the knapsack is smaller. This suggests that there are two arguments to the recursive problem: the set of items to chose from, and the available capacity of the knapsack.

- Notation: Let us denote by $optknapsack(k, w)$ the maximal value obtainable when filling a knapsack of capacity w using items among items 1 through k.

- To solve our problem we'll call $optknapsack(n, W)$.

- The overall strategy: The idea is to consider each item, one at a time. When we reach item k, we have two choices: either it's part of the optimal solution, or not. We need to compute both options, and chose the best one.

```
optknapsack(k, w)

    //basecase
    if (w ≤ 0): return 0
    if (k ≤ 0): return 0

    //choice 1: take item k in the backpack
    IF (weight[k] ≤ w): with = value[k] + optknapsack(k − 1, w − weight[k])
    ELSE: with = 0

    //choice 2: do not take item k in the backpack
    without = optknapsack(k − 1, w)

    //the optimal solution is the best of the two
    RETURN max { with, without }
```

- Correctness: Follows from optimal substructure. The algorithm considers all possible choices at every step, and takes the best one.
- **Running time analysis:** Note that the function optknapsack has two parameters, both k and w. Its running time is function of both.

  Let $T(n, W)$ be the running time of optknapsack(n, W). We have:

  $$T(n, W) = T(n-1, W) + T(n-1, W - w[n]) + \Theta(1)$$

  The worst case is when $w[i] = 1$ for all $1 \leq i \leq n$.

  $$T(n, W) = T(n-1, W) + T(n-1, W-1) + \Theta(1)$$

  Since $T(n-1, W) > T(n-1, W-1)$ we get that

  $$T(n, W) > T2(n-1, W-1)$$

  The recursion depth is $\min(n, W)$ steps, and at every step, the time doubles. Therefore $T(n, W) = \Omega(2^{\min(n,W)})$. This is exponential.
- **Why exponential?** How many different sub-problems are there? Answer: Each call to *optknapsack*() has a value for k and a value for w. There are n values for k and W values for w. In total $n \cdot W$ different sub-problems.

- Overlapping calls: Sketch the recursion tree for $n = 5$ and $w[1] = w[2] = w[3] = w[4] = 1$

DP recursive solution with memoization

- Idea: we'll use a table to store solutions to subproblems, which will prevent a subproblem optknapsack(k,w) to be computed more than once. Note that since a subproblem is of the form $optknapsack(k, w)$, i.e. has two arguments, the table must be two dimensional.
- **The table:** We create a table T of size $[1..n][1..W]$. Entry $T[i][w]$ will store the result of optknapsack(i, w).
- **Initialize:** First we initialize all entries in the table as 0 (in this problem we are looking for max values when all item values are positive, so 0 as initial value is safe).
- We modify the algorithm to check this table before launching into computing the solution.

```
optknapsackDP(k, w)
    //basecase
    if (w ≤ 0): return 0
    if (k ≤ 0): return 0
    //if solution already computed, return it
    IF (table[k][w] ≠ 0): RETURN table[k][w]
    //choice 1: take item k in the backpack (if possible)
    IF (w[k] ≤ w): with = v[k] + optknapsackDP(k − 1, w − w[k])
    ELSE: with = 0
    //choice 2: do not take item k in the backpack
    without = optknapsackDP(k − 1, w)
    //store solution in the table
    table[k][w] = max { with, without }
    RETURN table[k][w]
```

- **Analysis:** Ignoring the recursion, each subproblem is computed in $O(1)$ time. The recursion will fill in the table (in worst case: the entire table), with each entry being filled at most once. Therefore in total the recursion adds a factor of $O(n \cdot W)$ to the cost of computing each entry. Overall this will run in $O(n \cdot W)$ time as we fill each entry in the table at most once, and there are nW spaces in the table.

Dynamic programming: iterative solution

It is possible to eliminate the recursion by filling the table in such an order so that there is no need of recursion. To see this we simply look at the order in which recursion fills the table.

Question: In what order is the table filling up? Where are the base-cases in the table?
Answer: Table is filling up left to right and top-down.

```
optknapsackDP_iterative

    create table[0..n][0..W] and initialize all entries to 0

    for (k = 1; k < n; k++)

        for (w = 1; w < W; w++)
            with = v[k] + table[k-1][w-w[k]]
            without = table[k-1][w]
            table[k][w] = max { with, without }

    RETURN table[n][W]
```

Analysis: $O(n \cdot W)$ Note that this is the same as for the top-down recursive approach. Basically we just get rid of recursion and its overhead, but the asypmtotic complexity stays the same.

From finding the optimal cost to finding the set of items

Describe how to extend this solution in order to compute the set of items in the backpack (corresponding to the optimal value).